MATH 504 HOMEWORK 5

Due Monday, November 5.

Problem 1. *Let M be a countable transitive model of ZFC and $Add(\omega, 1)$ be the poset of all functions $f : \mathrm{dom}(f) \to \{0, 1\}$, where $\mathrm{dom}(f)$ is a finite subset of ω. Let G be $Add(\omega, 1)$-generic filter over M. As we did in class, define $f^* = \bigcup G$ and $a = \{n \mid f^*(n) = 0\}$. Recall that we proved in class that f^* is a total function with domain ω.*

(1) *Find two different $Add(\omega, 1)$-names in M for a, say σ and τ, such that $\sigma_G = \tau_G = a$.*
(2) *Show that in $M[G]$, a is an unbounded subset of ω.*

Problem 2. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that G is a $\mathbb{P}$-generic filter over M and $p \in G$.*

(1) *Suppose that $D \subset \mathbb{P}$ is such that for every $q \leq p$, there is $r \leq q$ with $r \in D$. Show that $G \cap D \neq \emptyset$. Such a set D is called dense below p.*
(2) *Let $A \subset \mathbb{P}$ be an antichain such that for every $q \in A$, $q \leq p$, and for every $r \leq p$, there is $q \in A$ such that r, q are compatible i.e. they have a common extension. Show that $G \cap A \neq \emptyset$. Such a set A is called a maximal antichain below p.*

Problem 3. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that $G \subset \mathbb{P}$ is a filter. A set $D \subset \mathbb{P}$ is called open dense if it is dense and whenever $q \leq p$ and $p \in D$, we have that $q \in D$. Show that G is generic if and only if G meets every open dense subset of $\mathbb{P}$.*

Problem 4. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that σ and τ are two $\mathbb{P}$-names in M, such that* $\mathrm{dom}(\sigma), \mathrm{dom}(\tau) \subset \{\check{n} \mid n < \omega\}$. *Let*

$$\pi = \{\langle \check{n}, p\rangle \mid (\exists q, r)(p \leq q \wedge p \leq r \wedge \langle \check{n}, q\rangle \in \sigma \wedge \langle \check{n}, r\rangle \in \tau)\}.$$

Show that $\pi_G = \tau_G \cap \sigma_G$ for any generic filter G over M.

Problem 5. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that σ is a $\mathbb{P}$-name in M, such that* $\mathrm{dom}(\sigma) \subset \{\check{n} \mid n < \omega\}$. *Let*

$$\pi = \{\langle \check{n}, p\rangle \mid (\forall q \in \mathbb{P})(\langle \check{n}, q\rangle \in \sigma \to q \perp p)\}.$$

Show that $\pi_G = \omega \setminus \sigma_G$ for any generic filter G over M.

Hint: show that $\{r \mid \exists p \geq r(\langle \check{n}, p\rangle \in \pi \vee \langle \check{n}, p\rangle \in \sigma)\}$ is dense.

Problem 6. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that τ is a $\mathbb{P}$-name in M. Let*

$$\pi = \{\langle \nu, p\rangle \mid \exists \langle \sigma, q\rangle \in \tau \exists r(p \leq r \wedge p \leq q \wedge \langle \nu, r\rangle \in \sigma)\}.$$

Show that $\pi_G = \bigcup \tau_G$ *for any generic filter* G *over* M.